Linear systems – Final exam – Solutions

Final exam 2022–2023, Tuesday 20 June 2022, 15:00 – 17:00

Problem 1 (?? points)

A simple model of a magnetic levitation system is given as

$$m\ddot{q}(t) = mg - \frac{1}{2}\frac{L}{(1+q(t))^2}u(t)^2, \tag{1}$$

with $q(t)$ the position of the levitated mass with mass $m > 0$ and $g > 0$ the gravitational constant. The input current to the electromagnet that suspends the mass is denoted by $u(t)$ and $L > 0$ is a constant.

(a) Write the system (1) in the form of a nonlinear state-space system $\dot{x} = f(x, u)$ by taking $x_1(t) = q(t)$ and $x_2(t) = \dot{q}(t)$.

(b) Let $\bar{x} = [\,\bar{q}\ 0\,]^{\mathrm{T}}$ be the desired equilibrium point for some $\bar{q} > 0$. Give the constant input $u(t) = \bar{u}$ with $\bar{u} > 0$ that achieves this equilibrium point.

(c) Linearize the state-space system around the equilibrium point given by $\bar{x}$ and $\bar{u}$.

Answer Problem 1 (a)

To write (1) in nonlinear state-space form, introduce the state

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}. \tag{2}$$

Then, it is immediate that $\dot{x}_1 = x_2$. The dynamics for x_2 follows from (1), leading to

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ g - \frac{1}{2m}\frac{L}{(1+x_1)^2}u^2 \end{bmatrix} = f(x, u). \tag{3}$$

Answer Problem 1 (b)

Let

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \bar{q} \\ 0 \end{bmatrix} \tag{4}$$

be the desired equilibrium point for some $\bar{q} > 0$. To find the constant input $u(t) = \bar{u}$, the equation

$$0 = f(\bar{x}, \bar{u}) \tag{5}$$

needs to be solved. Using (3), we obtain $0 = \bar{x}_2$ for the first coordinate, which indeed corresponds to (4). The equation for the second coordinate yields

$$g = \frac{1}{2m}\frac{L}{(1+\bar{q})^2}\bar{u}^2, \tag{6}$$

which has the solution (recall that a positive solution $\bar{u} > 0$ is sought)

$$\bar{u} = \sqrt{\frac{2mg}{L}}(1+\bar{q}). \tag{7}$$

Answer Problem 1 (c)

In order to find the linearized dynamics around the equilibrium point given by $\bar{x}$ and $\bar{u}$, define the perturbations

$$\tilde{x} = x - \bar{x}, \qquad \tilde{u} = u - \bar{u}. \tag{8}$$

Then, the linearized dynamics is given as

$$\dot{\tilde{x}}(t) = \frac{\partial f}{\partial x}(\bar{x}, \bar{u})\tilde{x}(t) + \frac{\partial f}{\partial u}(\bar{x}, \bar{u})\tilde{u}(t), \tag{9}$$

after which it can be concluded from (3) that

$$\frac{\partial f}{\partial x}(x, u) = \begin{bmatrix} 0 & 1 \\ \frac{L}{m}(1 + x_1)^{-3}u^2 & 0 \end{bmatrix}. \tag{10}$$

Evaluation of the result at $(\bar{x}, \bar{u})$ gives, after substitution of (7),

$$\frac{\partial f}{\partial x}(\bar{x}, \bar{u}) = \begin{bmatrix} 0 & 1 \\ 2g(1 + \bar{q})^{-1} & 0 \end{bmatrix}. \tag{11}$$

Similarly,

$$\frac{\partial f}{\partial u}(x, u) = \begin{bmatrix} 0 \\ -\frac{1}{m}\frac{L}{(1+x_1)^2}u \end{bmatrix} \tag{12}$$

such that

$$\frac{\partial f}{\partial u}(\bar{x}, \bar{u}) = \begin{bmatrix} 0 \\ -\sqrt{\frac{2gL}{m}\frac{1}{1+\bar{q}}} \end{bmatrix}. \tag{13}$$

Problem 2 (5 + 5 + 4 + 10 + 8 = 32 points)

Consider the linear system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}, \qquad \text{with} \quad A = \begin{bmatrix} -1 & 1 \\ -4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

and where $x(t) \in \mathbb{R}^2$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$.

(a) Is the system internally stable?

(b) Give the transfer function for the system.

(c) Verify that the system is controllable.

(d) Find a nonsingular matrix T and real numbers α_1, α_2 such that

$$TAT^{-1} = \begin{bmatrix} 0 & 1 \\ \alpha_1 & \alpha_2 \end{bmatrix}, \qquad TB = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

It is sufficient to give T^{-1}.

(e) Use the matrix T from (d) to design a state feedback controller $u(t) = Fx(t)$ such that the resulting closed-loop system satisfies $\sigma(A + BF) = \{-1, -3\}$.

Answer Problem 2 (a)

To determine internal stability, consider the characteristic polynomial of A as

$$\Delta_A(s) = \det(sI - A) = \begin{vmatrix} s+1 & -1 \\ 4 & s-2 \end{vmatrix} = (s+1)(s-2) + 4 = s^2 - s + 2, \tag{14}$$

and recall that the system is internally stable if and only if the polynomial $\Delta_A(s)$ is stable. However, $\Delta_A(s)$ is a polynomial of degree two and its coefficients do not all have the same sign. Hence, the system is not internally stable.

Alternatively, we may compute the eigenvalues directly by solving

$$0 = \Delta_A(\lambda) = \lambda^2 - \lambda + 2, \tag{15}$$

leading to the eigenvalues

$$\lambda_1 = \frac{1 + i\sqrt{7}}{2}, \qquad \lambda_2 = \frac{1 - i\sqrt{7}}{2} \tag{16}$$

using the quadratic formula. As both have positive real parts, the system is not internally stable.

Answer Problem 2 (b)

Recall that the transfer function $T(s)$ is given by

$$T(s) = C(sI - A)^{-1}B. \tag{17}$$

Therefore, we first compute

$$(sI - A)^{-1} = \frac{1}{\Delta_A(s)} \operatorname{adj}(sI - A) = \frac{1}{s^2 - s + 2} \begin{bmatrix} s-2 & 1 \\ -4 & s+1 \end{bmatrix} \tag{18}$$

and subsequently obtain

$$T(s) = \frac{1}{s^2 - s + 2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-2 & 1 \\ -4 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{s+1}{s^2 - s + 2}. \tag{19}$$

Answer Problem 2 (c)

To verify controllability, compute

$$\begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \tag{20}$$

and note that

$$\operatorname{rank} \begin{bmatrix} B & AB \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = 2 = n, \tag{21}$$

where n is the state-space dimension. Hence, the system is controllable.

Answer Problem 2 (d)

First, recall the characteristic polynomial (14) as

$$\Delta_A(s) = s^2 - s + 2 \tag{22}$$

and define

$$a_1 = -1, \quad a_0 = 2. \tag{23}$$

to write

$$\Delta_A(s) = s^2 + a_1 s + a_0. \tag{24}$$

Then, as the pair (A, B) is controllable, there exists a nonsingular matrix T such that

$$TAT^{-1} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, \qquad TB = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{25}$$

which is the controllability canonical form. By comparing this to the matrices in the question, this means that we can choose α_1 and α_2 such that

$$\alpha_1 = -a_0 = -2, \quad \alpha_2 = -a_1 = 1. \tag{26}$$

To find the matrix T that achieves the transformation, define the vector q_2 as

$$q_2 = B = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{27}$$

and vector q_2 as

$$q_1 = AB + a_1 B = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{28}$$

Now, define T through its inverse as

$$T^{-1} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}. \tag{29}$$

Using

$$T = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}, \tag{30}$$

a direct calculation shows that indeed

$$TAT^{-1} = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}, \quad TB = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{31}$$

verifying the desired result.

Answer Problem 2 (e)

As a first step, we define a polynomial $p(s)$ with roots at the desired eigenvalues for $A+BF$. This leads to

$$p(s) = (s+1)(s+3) = s^2 + 4s + 3, \tag{32}$$

with can be written as

$$p(s) = s^2 + p_1 s + p_0 \tag{33}$$

with

$$p_1 = 4, \qquad p_0 = 3. \tag{34}$$

Our objective is to find a matrix F such that

$$\Delta_{A+BF}(s) = p(s). \tag{35}$$

To achieve this, note that

$$\Delta_{A+BF}(s) = \Delta_{T(A+BF)T^{-1}}(s) = \Delta_{TAT^{-1}+TBFT^{-1}}(s). \tag{36}$$

Using the matrix T from problem (d), this gives

$$TAT^{-1} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, \quad TB = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{37}$$

Next, denote

$$FT^{-1} = \begin{bmatrix} f_0 & f_1 \end{bmatrix}, \tag{38}$$

such that

$$TAT^{-1} + TBFT^{-1} = \begin{bmatrix} 0 & 1 \\ f_0 - a_0 & f_1 - a_1 \end{bmatrix} \tag{39}$$

As this matrix is in companion form, we can easily obtain its characteristic polynomial as

$$\Delta_{TAT^{-1}+TBFT^{-1}}(s) = s^2 + (a_1 - f_1)s + (a_0 - f_0). \tag{40}$$

Now, after recalling (36), we see that the objective (35) is achieved if and only if

$$a_1 - f_1 = p_1, \qquad a_0 - f_0 = p_0, \tag{41}$$

which can readily be solved to obtain

$$f_1 = a_1 - p_1 = -1 - 4 = -5, \qquad f_0 = a_0 - p_0 = 2 - 3 = -1. \tag{42}$$

Finally, solving the linear equation

$$FT^{-1} = \begin{bmatrix} f_0 & f_1 \end{bmatrix} = \begin{bmatrix} -1 & -5 \end{bmatrix}, \tag{43}$$

leads to

$$F = \begin{bmatrix} -2 & -1 \end{bmatrix}. \tag{44}$$

Problem 3 (14 points)

Show that the matrix pair (A, B) is controllable if and only if the matrix pair $(A + BF, B)$ is controllable for any matrix F.

Hint. Use the Hautus test.

The *if* part is clear (just take $F = 0$), so we focus on the *only if*. Let the matrix pair (A, B) be controllable. Denote by n the number of rows (and columns) of A, i.e., $A \in \mathbb{R}^{n\times n}$, and let m be such that $B \in \mathbb{R}^{n\times m}$. By the Hautus test, we have that

$$\operatorname{rank}\begin{bmatrix} A - \lambda I & B \end{bmatrix} = n \tag{45}$$

for all $\lambda \in \sigma(A)$. Note that this is equivalent to requiring (45) *for all* $\lambda \in \mathbb{C}$, as $\operatorname{rank}(A - \lambda I) = n$ for all $\lambda \notin \sigma(A)$.

We give two possible approaches for finalizing the proof.

Approach 1. Let $F \in \mathbb{R}^{m\times n}$ be arbitrary and note that, for any $\lambda \in \mathbb{C}$,

$$\begin{bmatrix} A + BF - \lambda I & B \end{bmatrix} = \begin{bmatrix} A - \lambda I & B \end{bmatrix} \begin{bmatrix} I & 0 \\ F & I \end{bmatrix}. \tag{46}$$

After observing that the matrix

$$\begin{bmatrix} I & 0 \\ F & I \end{bmatrix} \tag{47}$$

is nonsingular, it follows that

$$n = \operatorname{rank}\begin{bmatrix} A - \lambda I & B \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A - \lambda I & B \end{bmatrix}\begin{bmatrix} I & 0 \\ F & I \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A + BF - \lambda I & B \end{bmatrix}, \tag{48}$$

for all $\lambda \in \mathbb{C}$, which shows that the matrix pair $(A + BF, B)$ is controllable.

Approach 2. Attempting to establish a contraction, assume that there exists $F \in \mathbb{R}^{n\times m}$ such that the matrix pair $(A + BF, B)$ is not controllable. By the Hautus test, this means that there exists a $\lambda \in \sigma(A + BF)$ such that

$$\operatorname{rank}\begin{bmatrix} A + BF - \lambda I & B \end{bmatrix} < n. \tag{49}$$

Equivalently, there exists $v \in \mathbb{C}^n$ with $v \neq 0$ such that

$$v^{\mathrm{T}}\begin{bmatrix} A + BF - \lambda I & B \end{bmatrix} = 0. \tag{50}$$

However, this implies that $v^{\mathrm{T}}B = 0$, such that also

$$v^{\mathrm{T}}\begin{bmatrix} A - \lambda I & B \end{bmatrix} = 0, \tag{51}$$

which contradicts (45). Hence, the matrix pair $(A + BF, B)$ is controllable for any F.

Problem 4 (20 points)

Consider a linear system (A, B, C, D) and denote by $y(t; u)$ the output response for zero initial conditions and $u : [0, \infty) \to \mathbb{R}^m$, i.e.,

$$y(t; u) = \int_0^t Ce^{A(t-\tau)} Bu(\tau) \,\mathrm{d}\tau + Du(t).$$

Note that this can be written, for any s such that $0 \leq s \leq t$, as

$$y(t; u) = \int_0^s Ce^{A(t-\tau)} Bu(\tau) \,\mathrm{d}\tau + \int_s^t Ce^{A(t-\tau)} Bu(\tau) \,\mathrm{d}\tau + Du(t). \tag{52}$$

Let the system be externally stable. Use (52) to prove that, if $u(t)$ is such that $\lim_{t\to\infty} u(t) = 0$, then

$$\lim_{t\to\infty} y(t; u) = 0.$$

Hint. Introduce the function $\mu : [0, \infty) \to \mathbb{R}$ defined as $\mu(t) = \sup\{|u(\tau)| : \tau \geq t\}$, where $|\cdot|$ denotes the Euclidean norm. You may use the fact that μ is decreasing and satisfies $\lim_{t\to\infty} \mu(t) = 0$.

First, note that external stability implies that

$$\int_0^\infty \|Ce^{At} B\| \,\mathrm{d}t < \infty. \tag{53}$$

To show that

$$\lim_{t\to\infty} y(t; u) = 0,$$

holds, we consider the Euclidean norm of $y(t; u)$ and obtain

$$\begin{aligned} |y(t; u)| &= \left| \int_0^s Ce^{A(t-\tau)} Bu(\tau) \,\mathrm{d}\tau + \int_s^t Ce^{A(t-\tau)} Bu(\tau) \,\mathrm{d}\tau + Du(t) \right| \\ &\leq \left| \int_0^s Ce^{A(t-\tau)} Bu(\tau) \,\mathrm{d}\tau \right| + \left| \int_s^t Ce^{A(t-\tau)} Bu(\tau) \,\mathrm{d}\tau \right| + \big|Du(t)\big|, \end{aligned} \tag{54}$$

using the triangle inequality. Next, the first term on the right-hand side can be bounded as

$$\left| \int_0^s Ce^{A(t-\tau)} Bu(\tau) \,\mathrm{d}\tau \right| \leq \int_0^s \big|Ce^{A(t-\tau)} Bu(\tau)\big| \,\mathrm{d}\tau \leq \int_0^s \big\|Ce^{A(t-\tau)} B\big\| |u(\tau)| \,\mathrm{d}\tau, \tag{55}$$

where we have used the definition of the matrix norm $\|\cdot\|$. Note that, for $\tau \geq 0$, we have $|u(\tau)| \leq \mu(0)$ by definition of the function μ, such that

$$\begin{aligned} \left| \int_0^s Ce^{A(t-\tau)} Bu(\tau) \,\mathrm{d}\tau \right| \leq \mu(0) \int_0^s \big\|Ce^{A(t-\tau)} B\big\| \,\mathrm{d}\tau &= \mu(0) \int_{t-s}^t \big\|Ce^{Ar} B\big\| \,\mathrm{d}r \\ &\leq \mu(0) \int_{t-s}^\infty \big\|Ce^{Ar} B\big\| \,\mathrm{d}r. \end{aligned} \tag{56}$$

Here, we have used the change of variables $r = t - s$ to obtain the equality. Note that the final expression is well-defined due to (53).

Following a completely analogous reasoning, noting that $|u(\tau)| \leq \mu(s)$ for $\tau \geq s$, we can bound the second term on the right-hand side of (54) as

$$\left| \int_s^t Ce^{A(t-\tau)} Bu(\tau) \,\mathrm{d}\tau \right| \leq \mu(s) \int_0^{t-s} \big\|Ce^{Ar} B\big\| \,\mathrm{d}r \leq \mu(s) \int_0^\infty \big\|Ce^{Ar} B\big\| \,\mathrm{d}r \tag{57}$$

Now, choose $s = \frac{1}{2}t$. Then, from (56) we obtain

$$\lim_{t\to\infty} \left| \int_0^{\frac{t}{2}} Ce^{A(t-\tau)} Bu(\tau)\,\mathrm{d}\tau \right| \leq \lim_{t\to\infty} \mu(0) \int_{\frac{t}{2}}^{\infty} \left\| Ce^{Ar} B \right\| \mathrm{d}r = 0, \tag{58}$$

whereas (57) leads to

$$\lim_{t\to\infty} \left| \int_{\frac{t}{2}}^{t} Ce^{A(t-\tau)} Bu(\tau)\,\mathrm{d}\tau \right| \leq \lim_{t\to\infty} \mu\big(\tfrac{1}{2}t\big) \int_0^{\infty} \left\| Ce^{Ar} B \right\| \mathrm{d}r = 0, \tag{59}$$

due to the properties of the function μ and (53).

The use of (58) and (59) in (54) leads to

$$\lim_{t\to\infty} |y(t;u)| \leq 0, \tag{60}$$

where we have also used that $\lim_{t\to\infty} |Du(t)| = 0$. This proves the desired result.

(10 points free)